



Finite-time exergoeconomic performance bound for a quantum Stirling engine

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Abstract

The finite-time exergoeconomic optimal performance for a quantum Stirling engine is studied. The maximum exergoeconomic profit, the optimal thermal efficiency and the optimal power output of the quantum Stirling engine are obtained. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Quantum Stirling engine; Exergoeconomic performance; Finite-time thermodynamics; Quantum statistical mechanics

1. Introduction

Stirling engine is an external combustion engine with brilliant prospects. It utilizes the resources of low price and easily obtainable energy such as solar, atomic, geothermal, industrial waste heat and marsh-gas etc. It has very important significance for environmental protection and moderating the tense petroleum needs in the world. With this great potential, the Stirling engine has been used widely in energy resource utilization and power engineering. Some authors have studied the performance optimization problem for the Stirling engine using the method of finite-time thermodynamic analysis [1,2]. Significant results have been obtained by Wu [3], Hu [4] and Ladas and Ibrahim [5]. However, in the work mentioned above, the classical engine with pure thermodynamic performance objective was studied while the quantum performance of engine's working fluid was not considered. The dynamical semi-group approach of the quantum theory of open systems has been introduced into finite time thermodynamics only recently [6–12]. Based on the Salamom and Nitzan's work [13] in which the economic index of profit was taken as the

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optimization objective, Chen [14,15], Wu [16] and DeVos [17] developed the method which is called finite-time exergoeconomic analyses or endoreversible thermoeconomics. The exergoeconomic optimum performance of a quantum-mechanical Stirling engine is studied in the present paper by using the modern non-equilibrium quantum statistical theory and the methods used in Refs. [6–17]. The maximum profit and the optimal thermal efficiency of the engine at the classical limit are derived. The results provide a solid theoretical basis for design and operation of Stirling engine.

2. Quantum-mechanical Stirling engine

The working fluid of the engine consists of many non-interacting harmonic oscillators or many non-interacting two-level systems. The former case will be carried out in this paper. The latter case has been analogously studied by Feldmann [11]. The population of the oscillators can be obtained from the Bose–Einstein distribution [12,18,19].

$$n = 1/[\exp(\beta\omega) - 1]. \quad (1)$$

where $\omega > 0$ is the frequency of the oscillator and $\beta = 1/T$ (T is the internal temperature of the working fluid).

We assume that the engine operates between an isothermal heat source B_1 at a temperature T_h and a heat sink B_2 at T_c . The endoreversible quantum-mechanical Stirling engine is made of two isothermal branches connected by two regenerating branches (see Fig. 1). Because of the heat transfer effect, the internal temperatures of the working fluid are T_1 and T_2 in the two isothermal processes, e.g. 1–2 and 3–4 in Fig. 1. The second law of thermodynamics requires $\beta_c > \beta_2 > \beta_1 > \beta_h$, where $\beta_i = 1/T_i$ ($i = h, c, 1$ and 2).

In the regenerating processes, the two pistons of the Stirling engine move simultaneously with the same velocity and direction. The volume of the working fluid is therefore invariant. The quantum working fluid flows through the regenerator and exchanges heat with the filler inside the regenerator. The regenerator has a high frequency and a short period. The population n of the harmonic oscillators increases drastically with the decrease in the reciprocal (β) of temperature. Hence, the oscillator frequency in the regenerating processes is approximately constant. In Fig. 1, the frequencies of the two regenerating processes connecting the two isotherms are given by $\omega = \omega_1$ and $\omega = \omega_2$ with $\omega_1 > \omega_2$. The cycle of the engine is illustrated in the (ω, n) plane. In the quantum calculation, we set $h' = h/2\pi = 1$, where h is the Planck's constant.

3. Cycle period

The Hamiltonian of the harmonic oscillator system S is described in the following form [8,12]

$$\mathbf{H}_S = \omega \mathbf{N} = \omega(\mathbf{a}^+ \mathbf{a}), \quad (2)$$

where $\mathbf{N}$ is the number operator, $\mathbf{a}^+$ and $\mathbf{a}$ are the Bosonic creation and annihilation operators $\{\mathbf{N} = (\mathbf{a}^+ \mathbf{a}); [\mathbf{a}, \mathbf{a}^+] = 1\}$.

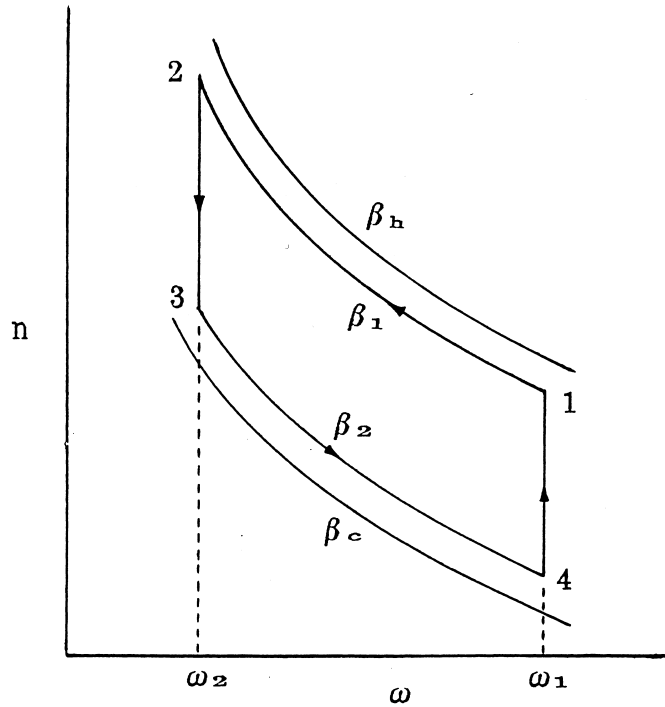


Fig. 1. Endoreversible quantum Stirling cycle in (n, ω) plane.

In the isothermal processes, the working fluid S becomes an open system owing to the coupling action between S and the heat source B_1 (or heat sink B_2). Using the generalized master equation in an open system and in Heisenberg picture, one obtains the motion equation of the operator $\mathbf{x}$ in the system S [20] as shown in the following equation.

$$d\mathbf{x}/dt = i\mathbf{L}_f\mathbf{x} - \sum \{(\omega_{kj})^+[\mathbf{x}, \mathbf{Q}_k]\mathbf{Q}_j - (\omega_{jk})^-\mathbf{Q}_j[\mathbf{x}, \mathbf{Q}_k]\}, \quad (3)$$

where $(\omega_{kj})^+$ and $(\omega_{jk})^-$ are constants related to the matrix element of the heat source B_1 (or heat sink B_2) density matrix, $\mathbf{Q}_j$ and $\mathbf{Q}_k$ are the operators of the working fluid system S , $[\cdot, \cdot]$ is the quantum Poisson's bracket, while

$$\mathbf{L}_f = \mathbf{L}_S + T_{\gamma B}\mathbf{L}_{SB}\rho_B, \quad (4)$$

where $\mathbf{L}_S$ is the Liouville's operator of the system S , $T_{\gamma B}$ represents the trace of the operator of the heat reservoir (B_1 or B_2), $\mathbf{L}_{SB}$ represents the coupling action between the working fluid S and the heat reservoir (B_1 or B_2), ρ_B is the density matrix of the heat reservoir (B_1 or B_2). Because the heat reservoir (B_1 or B_2) is very large and its internal relaxation is very strong, the disturbance of the working fluid on the reservoir can be neglected. Thus we obtain

$$T_{\gamma B}\mathbf{L}_{SB}\rho_B = 0. \quad (5a)$$

Using the definition of Louville's operator for the number operator $\mathbf{N}$, one obtains

$$\mathbf{L}_S \mathbf{N} = [\mathbf{H}_S, \mathbf{N}] = [\omega \mathbf{N}, \mathbf{N}] = 0. \quad (5b)$$

Combining Eqs. (4), (5a) and (5b) gives

$$\mathbf{L}_f \mathbf{N} = (\mathbf{L}_S + T_{\gamma B} \mathbf{L}_{SB} \rho_B) \mathbf{N} = 0. \quad (6)$$

Setting $\mathbf{x} = \mathbf{N}$, $\mathbf{Q}_1 = \mathbf{a}^+$, $\mathbf{Q}_2 = \mathbf{a}$ and combining Eqs. (3) and (6) gives

$$dn/dt = d\mathbf{N}/dt = -[(\omega_{12})^+ + (\omega_{12})^-] - [(\omega_{21})^+ + (\omega_{21})^-]n + [(\omega_{21})^+ + (\omega_{21})^-], \quad (7a)$$

where $\mathbf{N} = n$ is the expectation values of the number operator $\mathbf{N}$, $[(\omega_{12})^+ + (\omega_{12})^-]$ and $[(\omega_{21})^+ + (\omega_{21})^-]$ are constants. The solution of Eq. (7a) is given by

$$n = n_e + (n_o - n_e) \exp\{-\{[(\omega_{12})^+ + (\omega_{12})^-] - [(\omega_{21})^+ + (\omega_{21})^-]\}t\}, \quad (8)$$

where $n_e = [(\omega_{21})^+ + (\omega_{21})^-] / \{[(\omega_{12})^+ + (\omega_{12})^-] - [(\omega_{21})^+ + (\omega_{21})^-]\}$ is the asymptotic stationary value of n , n_o the value of n at $t=0$. Eq. (8) is similar to Eqs. (3) and (4) of Ref. [8]. Using the same method as used in Ref. [8], one obtains $[(\omega_{21})^+ + (\omega_{21})^-] = a[\exp(q\beta_j\omega)]$ and $[(\omega_{12})^+ + (\omega_{12})^-] = a\{\exp[(l+q)\beta_j\omega]\}$. Substituting them into Eq. (7a) yields

$$dn/dt = -a[\exp(q\beta_j\omega)]\{\exp(\beta_j\omega) - 1\}n - 1\}, \quad (7b)$$

where a and q are constants, $T_j = 1/\beta_j$ ($j = h, c$) are the temperatures of the heat reservoirs, $n = \mathbf{N}$ is given by Eq. (1).

Substituting Eq. (1) into Eq. (7b), expanding the time integral and taking the second order in $\beta\omega$ and $\beta_j\omega$ at the classical limit ($n \gg 1$ and $\beta\omega \ll 1$) gives

$$t_1 = \int [(\mathbf{d}n/\mathbf{d}\omega)/(\mathbf{d}n/\mathbf{d}t)]\mathbf{d}\omega = A/(\beta_1 - \beta_h), \quad \omega_1 < \omega < \omega_2. \quad (9)$$

$$t_2 = A/(\beta_c - \beta_2), \quad (10)$$

where t_1 and t_2 are the isothermal processes times of processes 1–2 and 3–4, $A = [(\omega_2)^{-1} - (\omega_1)^{-1}]/a$ is a positive constant in matters of the quantum performance of the working fluid.

Assuming the times of the regenerating processes 2–3 and 4–1 are t_3 and t_4 , respectively, the cycle period time, τ , can be given by the following expression

$$\tau = t_1 + t_2 + t_3 + t_4. \quad (11a)$$

Let $b = (t_1 + t_2)/\tau$ denote the ratio of the isothermal time to the total cycle period time, one obtains

$$\tau = (t_1 + t_2)/b = A[(\beta_1 - \beta_h)^{-1} + (\beta_c - \beta_2)^{-1}]/b. \quad (11b)$$

4. Maximum profit

From Eq. (2) the rate of change of energy of the working fluid is obtained

$$dE/dt = d(\mathbf{H}_S)/dt = (d\omega/dt)n + \omega(dn/dt). \quad (12)$$

Comparing Eq. (12) with the time derivative of the first law of thermodynamics, we have the instantaneous power and the instantaneous heat flow, respectively:

$$dW/dt = -(d\omega/dt)n, \quad (13)$$

$$dQ/dt = (d\omega/dt)n. \quad (14)$$

Substituting Eq. (1) into Eq. (14) and expanding the heat quantity integral to first order in $\beta_i\omega$ ($i = h, 1$ or $c, 2$) gives

$$Q_1 = \int \omega dn = \lambda/\beta_1, n_1 < n < n_2, \quad (15)$$

$$Q_2 = \lambda/\beta_2, \quad (16)$$

where Q_1 and Q_2 are the absorbing heat quantity in the isothermal process 1–2 and the exothermic quantity in the isothermal process 3–4, respectively, and the positive constant $\lambda = \ln(\omega_1/\omega_2)$ is the natural logarithm of the frequency ratio.

The heat exchange of processes 2–3 and 4–1 are carried out inside the engine. If the regenerator is free of loss, the quantity of the regeneration is independent of the energy dissipation of the cycle. Thus the output work and the thermal efficiency of the cycle are

$$W = Q_1 - Q_2 \quad (17)$$

and

$$\eta = (Q_1 - Q_2)/Q_1 = 1 - x, \quad (18)$$

where $x = \beta_1/\beta_2$.

In order to obtain the optimal economic performance we must study the profit of the engine. The profit is defined as the difference between the revenue rate of the work output and the cost rate of the exergy input [13–16]. The exergy input of the cycle is equal to the reversible work of the cycle with the same working condition. The reversible work produced by the engine with the same working condition is

$$W_r = Q_1(1 - T_o/T_h) - Q_2(1 - T_o/T_c) = \epsilon_1 Q_1 - \epsilon_2 Q_2, \quad (19)$$

where $T_o = 1/\beta_o$ is the ambient temperature, $\epsilon_1 = 1 - T_o/T_h = 1 - \beta_h/\beta_o$ and $\epsilon_2 = 1 - T_o/T_c = 1 - \beta_c/\beta_o$ are the Carnot coefficient of absorbing heat and removing heat, respectively.

We assume that the prices of exergy input and work output are ϕ_1 and ϕ_2 . The profit of the engine is

$$\Pi = (\phi_2 W - \phi_1 W_r)/\tau. \quad (20a)$$

Substituting Eqs. (11b), (15), (16) and (19) into Eq. (20a) yields

$$\Pi = (\lambda b \phi_2 / A) [(1 - r\epsilon_1)/x - (1 - r\epsilon_2)] [\beta_2 / (\beta_2 x - \beta_h) + \beta_2 / (\beta_c - \beta_2)]^{-1}, \quad (20b)$$

where $r = \phi_1 / \phi_2$ is the price ratio. In order to assure the engine be profitable, the price ratio r should satisfy the relation of $0 < r < 1$.

Using Eqs. (20a) and (20b) and the theory of extreme value of a binary function, the maximum profit is obtained as follows

$$\Pi_m = (\lambda b \phi_2 / A) (1 - r\epsilon_2) [(f - \epsilon_3) / (1 + \epsilon_3)]^2, \quad (21)$$

where $f = [(1 - r\epsilon_1) / (1 - r\epsilon_2)]^{1/2}$, $\epsilon_3 = (\beta_h / \beta_c)^{1/2} = (T_c / T_h)^{1/2}$. The conditions at the extreme point corresponding to Π_m are:

$$\beta_{2o} = \beta_c (1 + \epsilon_3) / (1 + f), \quad (22)$$

$$x_o = f \epsilon_3. \quad (23)$$

The corresponding optimal working temperatures of the working fluid in the isothermal processes are:

$$T_{2o} = 1 / \beta_{2o}, \quad (24)$$

$$T_{1o} = 1 / (\beta_{2o} x_o). \quad (25)$$

5. Discussion

(1) *Optimal thermal efficiency.* The thermal efficiency of the engine that corresponds to the maximum profit is

$$\eta_o = 1 - x_o = 1 - \epsilon_3 [(1 - r\epsilon_1) / (1 - r\epsilon_2)]^{1/2}, \quad (26)$$

where η_o is the thermal efficiency of the quantum-mechanical Stirling engine for the finite-time exergeoeconomic performance bound at the classical limit.

(2) *Optimal power output.* The optimal power output of the engine corresponding to the maximum profit is

$$P_o = (W/\tau)_o \quad (27a)$$

or

$$P_o = (\lambda b / A) (1 - f \epsilon_3) (1 - \epsilon_3 / f) (1 + \epsilon_3)^{-2}. \quad (27b)$$

When the power output is P_o , the engine has the optimal economic benefit and its profit has the maximum value Π_m .

(3) *Influence of the price ratio.* The price ratio r directly affects the thermodynamic and economic performances of the engine. With the typical parameters of working conditions $T_h = 1000$ K, $T_c = 290$ K, $T_o = 300$ K and $\lambda b/A = \lambda b\phi_2/A = 1$, the relationships of $\Pi_m - r$, $P_o - r$ and $\eta_o - r$ are shown in Figs. 2 and 3, respectively.

When the price of the work output is far higher than that of the exergy input, $r = \phi_1/\phi_2 \rightarrow 0$, Eqs. (26), (27b) and (21)) become

$$\eta_o = \eta_{CA}, \quad (28)$$

$$P_o = (\lambda b/A)(\eta_{CA})^2/(2 - \eta_{CA})^2, \quad (29)$$

$$\Pi_m = (\lambda b\phi_2/A)(\eta_{CA})^2/(2 - \eta_{CA})^2 = \phi_2 P_o, \quad (30)$$

where $\eta_{CA} = 1 - \epsilon_3 = 1 - (\beta_h/\beta_c)^{1/2} = 1 - (T_c/T_h)^{1/2}$ is the C–A (Curzon–Ahlborn [21]) efficiency. Eq. (30) indicates that in the limit $r \rightarrow 0$, the maximum profit index of the engine is transformed into the maximum power output index and the performance bound is the C–A efficiency η_{CA} .

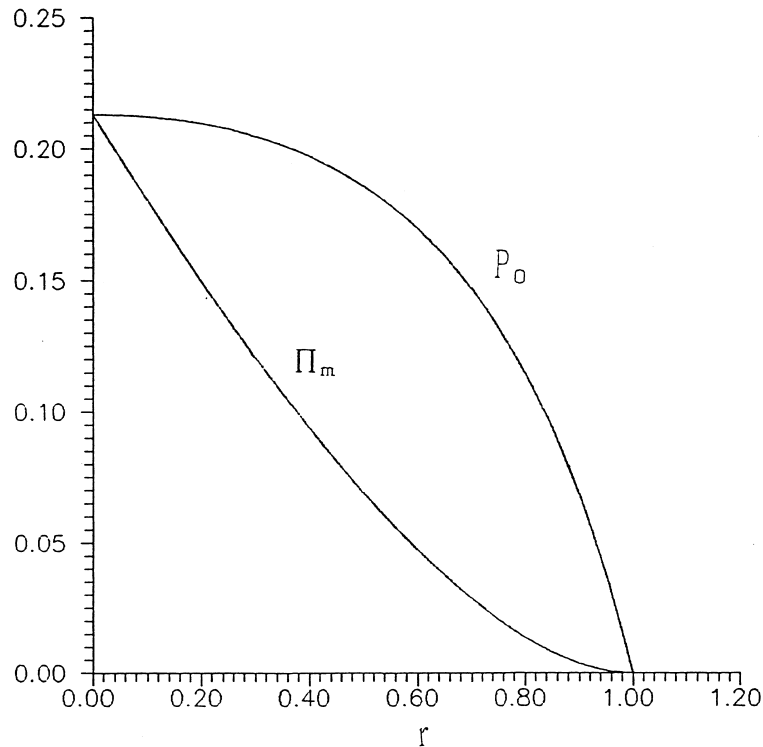


Fig. 2. Relationship among optimal power output (P_o), maximum profit (Π_m) and price ratio (r).

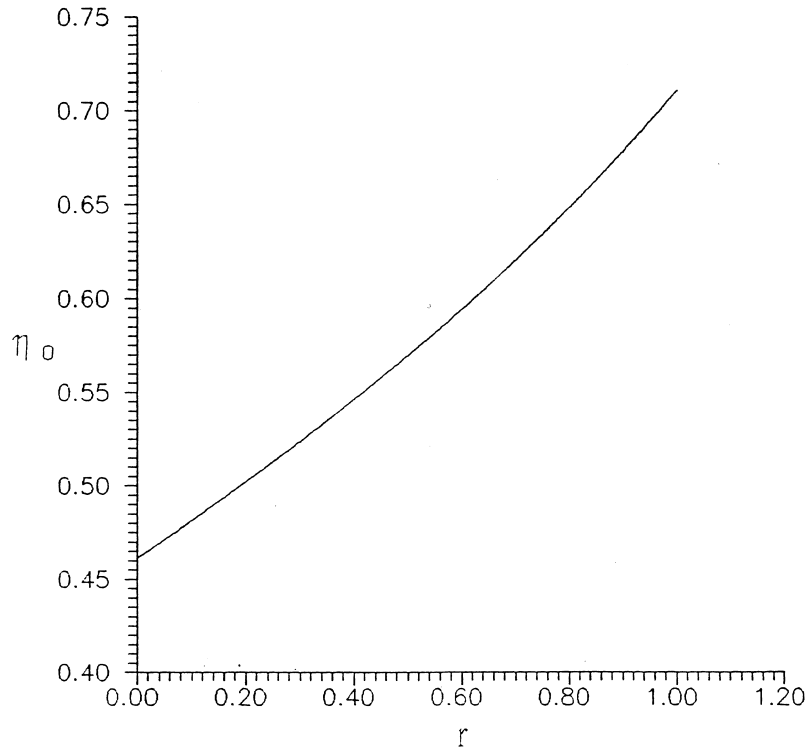


Fig. 3. Relationship between optimal efficiency (η_o) and price ratio (r).

When $r = \phi_1/\phi_2 \rightarrow 0$, Eqs. (26), (27b) and (21) become

$$\eta_o = \eta_C, \quad (31)$$

$$P_o = 0, \quad (32)$$

$$\Pi_m = 0, \quad (33)$$

where $\eta_C = 1 - \beta_h/\beta_c = 1 - T_c/T_h$ is the Carnot efficiency corresponding to a classical reversible Stirling cycle.

In general case of $0 < r < 1$, the optimal efficiency of the engine satisfies

$$\eta_{CA} \leq \eta_o < \eta_C. \quad (34)$$

(4) *Influence of the quantum performance of the working fluid.* In Eqs. (21), (26) and (27b) the constants A and λ are functions of the oscillator's frequency ω_1 and ω_2 . They depend on the quantum performance of the working fluid. Therefore, under given working condition, choosing a suitable quantum working fluid to enlarge the constant λ/A is very helpful for improving the thermodynamic and economic performance of the engine.

6. Conclusion

Economics is a major index for a real thermal power cycle. Combining finite-time thermodynamics and non-equilibrium quantum statistical mechanics with exergoeconomics, this paper derives the maximum profit and its corresponding performance bound (η_o) of an endoreversible quantum Stirling engine. The result of the paper provide a profit bound for designing a real Stirling engine whose working fluid is a quantum fluid, such as He, Ne, D₂ etc. In the work of Feldman et al. [11], the cycle of operation is analogous to a four-stroke Otto cycle and the work output was taken as optimization objective. The major advances of this paper over Refs. [6–12] are: (1) the modeling of a quantum Stirling engine with working medium consisting of non-interacting harmonic oscillators, and (2) the introduction of a profit objective. A future step of this analysis is to extend the working fluid model from the harmonic oscillators or two-level system to a three-level amplifier as done by Geva and Kosloff [9,10].

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